Exercise 34

At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

Solution

The relationship between velocity and position in one dimension is

$$v = \frac{dx}{dt}.$$

For the vectors here, it becomes

$$\mathbf{F} = \frac{d\mathbf{X}}{dt}.$$

We don't know what the position is, so we'll approximate the derivative by the difference quotient.

$$\mathbf{F} \approx \frac{\mathbf{X}(t) - \mathbf{X}(t_0)}{t - t_0}$$

Evaluate the velocity at the particle's initial position, $x = 1$ and $y = 3$, and plug in $t = 1.05$ and $t_0 = 1$ on the right side.

$$\mathbf{F}\Big|_{\substack{x=1 \\ y=3}} \approx \frac{\mathbf{X}(1.05) - \mathbf{X}(1)}{1.05 - 1}$$

$$\langle 1, -1 \rangle \approx \frac{\mathbf{X}(1.05) - \langle 1, 3 \rangle}{0.05}$$

Solve for $\mathbf{X}(1.05)$, the position vector of the particle at $t = 1.05$.

$$\langle 0.05, -0.05 \rangle + \langle 1, 3 \rangle \approx \mathbf{X}(1.05)$$

Therefore,

$$\mathbf{X}(1.05) \approx \langle 1.05, 2.95 \rangle.$$